MTH1302 2nd Summary

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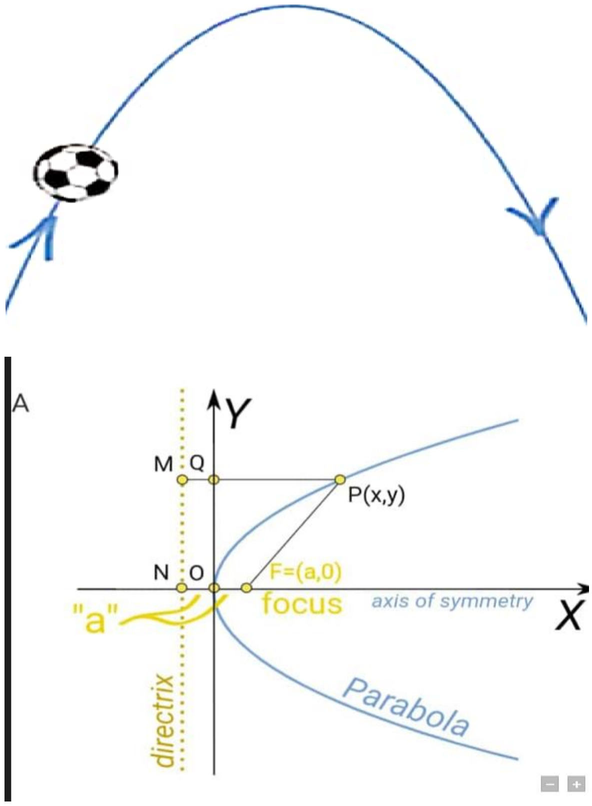
PARABOLA

Definition: a parabola is a curve where a point is at equal distance from:

- A fixed point (the focus) and
- A fixed straight line (the directrix)

A simple real life situation with a parabola is the curve/arc that a kicked football makes in the atmosphere.

{Image 1}



- The directrix as said above previously is a straight line at a fixed position on a parabola.
- The focus – as also said earlier is a fixed point on the parabola
- The axis of symmetry – this one goes through the focus at right angles to the direction
- The vertex – it is halfway between the focus and directrix. It is the point where the parabola meets its largest turn.

How to sketch a parabola

To do a sketch of a parabola,

a) Find the vertex; to get the vertex, we consider the type of the equation of the parabola

   i. First form: $(f(x) = a(x-h)^2 + k)$: the two pieces of information that we get here are that if a is positive$(+a)$, the parabola will open up and if negative$(-a)$, the parabola will open down.

      Secondly the vertex of the parabola is the point (h, k). Be very careful of the signs when getting the vertex.

   ii. Second Form: $f(x) = (ax^2 + bx + c)$:This form is the general form of the parabola. As was said earlier, the sign of 'a' determines the direction of the opening of the parabola. To get the vertex for this general from we use$(-b/2a, F(-b/2a))$. i.e we compute the x coordinate from a and b and then plug into the function so as to get the y coordinate.

b) Find the y-intercept (0, f(0)) , we simply put the values of 'x' in the given equation or the function of x , i.e : $F(x) = ax^2 + b(x) + c$
   Assuming x=0, then y intercept

   $F(0) = a(0)2 + b(0) + c = 0 + 0 + c = c.$

   Therefore (0, c) the y intercept equals c.

c) Solve f(x) = 0 – this implies finding the coordinate of the x intercept if they exist.
d) Make sure that at least one point to either side of the vertex is available
e) Sketch the graph.

Example 1

Sketch the graph of the parabola

$f(x) = 2(x+3)^2 - 8$

Solution

$f(x) = 2(x+3)^2 -8$

The equation is of the form $f(x) = a(x-b)^2 + K$

To find the vertex (h, k) i.e h=-3 and k=-8, vertex = (-3,-8). a = 2 i.e the parabola opens up

Next, we get the intercept of y intercept i.e $(0, f(0))$ i.e $x = 0, y = ?$

$f(0) = 2(0+3)2 - 8 = 2(9) - 8 = 10$ Therefore y intercept = (0, 10)

Next we find the next intercept

F(x) = 0 – $2(x+3)^2$ -8 = 0

Therefore a = 2

(x+3) = 8/2= 4

(x+3) = √4 = ±2

x = -3 ± 2 therefore x =-1, x = -3

Therefore x has two intercepts as (-5, 0) and (-1,0) and y=0

Now at this point, we have got points on either side of the vertex so we are officially done with finding the points.

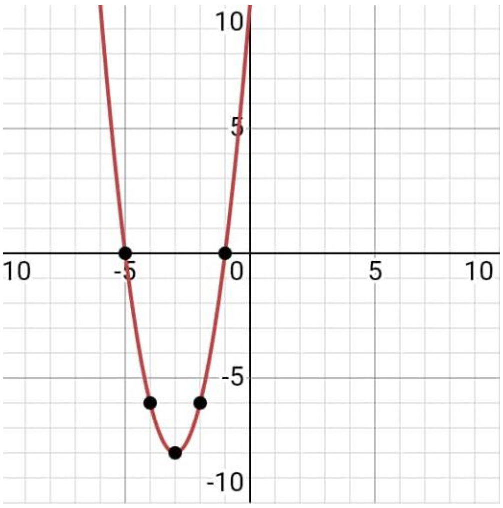
To find the second point using the y intercept and the axis of symmetry, notice first that the y intercept has as x coordinate of -3 .This means that the y intercept is a distance of 3 to the right of the axis of symmetry since that will move straight up from the vertex.

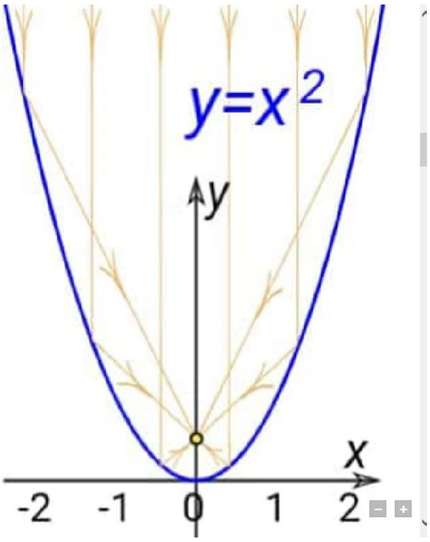
Now the left part of the graph will be a mirror image of the right part. So, since there is a point at y=10 that is a distance of 3 to the right of the axis of symmetry, there must also be a point at y =10, that is a distance of 3 to the left of the axis of symmetry.

Since the x coordinate of the vertex is -3 and this new point is a distance of 3 to the left, its 'x' coordinate must be -6. Then the coordinates of this new points ae (-6,10). We can verify by $f(x) = a(x-b)^2 + k$ but x = -6, a = 2 , b = -3, k=-8

Therefore $f(-6)= 2(-6+3)^2 -8 = 10$

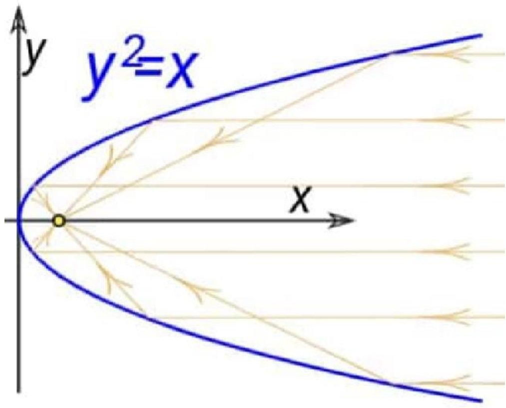
The graph of $f(x) = 2(x+3)^2 -8$ is shown below





Equations of a parabola

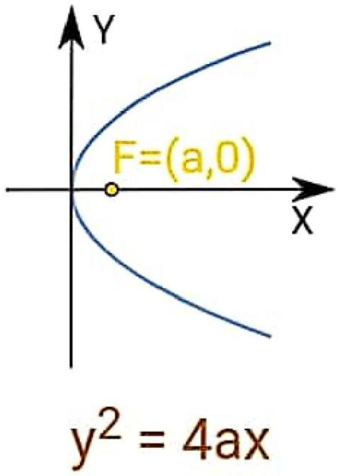
The simple equation is $y = x^2$



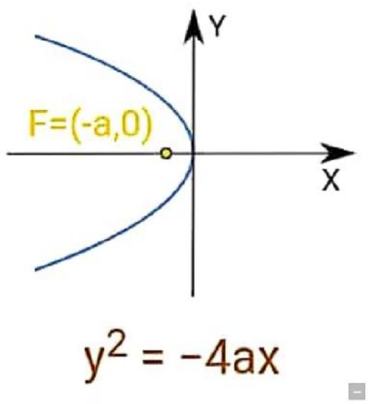
Turned on its side, it becomes $y^2 = x$

In the general form $y^2 = 4ax$. Where 'a' is the distance from the origin to the focus (and also from the origin to the direction). We have different forms of the equation as follows:

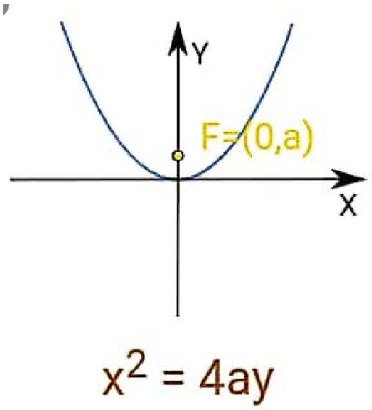
i. $y^2 = 4ax$



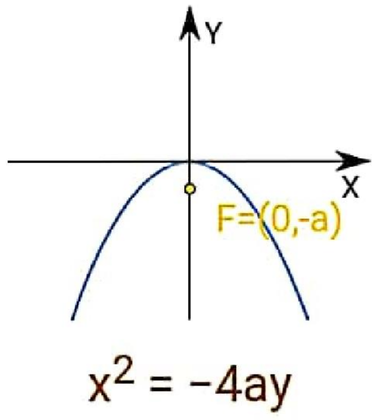
ii. $y^2 = -4ax$



iii. $x^2 = 4ay$



iv. $x^2 = -4ay$



Example 2

Find the focus for the equation $y^2 = 5x$

<u>Solution</u>

$y^2 = 5x$

but $y^2 = 4ax$

therefore $y^2 = 4(5/4)x = a\ 5/4$

therefore focus of $y^2 = 5x$ is F = (a,0)

= (5/4, 0)

Exercise

Sketch the graphs of the following parabolas –

a) $F(x) = -(x-2)^2 - 1$
b) $C(x) = 3(x+1)^2 - 6$
c) $H(x) = x^2 + 4$

References:

- Paul Dawkins (copyright 2003 to 2019)
- Webpage: mathsisfun.com (copyright 2018)
- https://www.mathsisfun.com/geometry/parabola.html

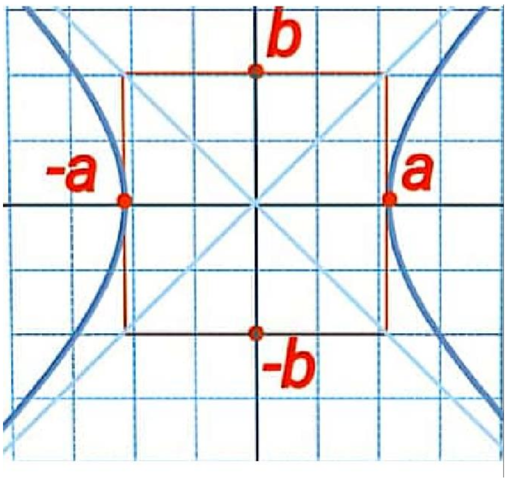
Hyperbola

The hyperbola is two curves that are like infinite bows. Each bow is called a branch.

In analytical geometry, the hyperbola is a comic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected.

Every hyperbola has two of symmetry. One of them is a line segment that passes through the center of the hyperbola and has vertices as its end points, and it is the traverse axis. The other is perpendicular to the traverse axis and has the co-vertices as its end points, and this is the conjugate axis.

The center of the hyperbola is the midpoint of both the axes i.e the point of intersection.



Equation of a hyperbola:

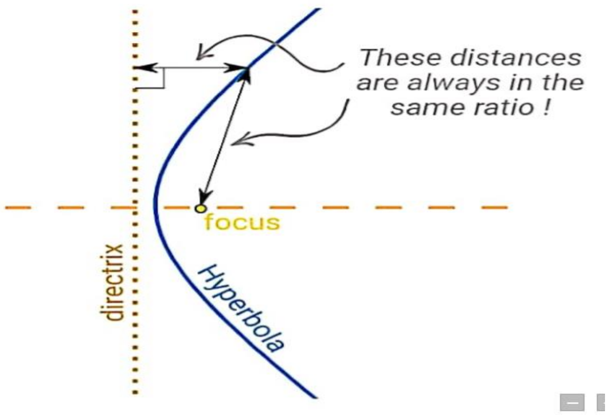
The equation of the curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

One vertex is at (a,0) and the other is at (-a,0).

The asymptotes are the straight lines:

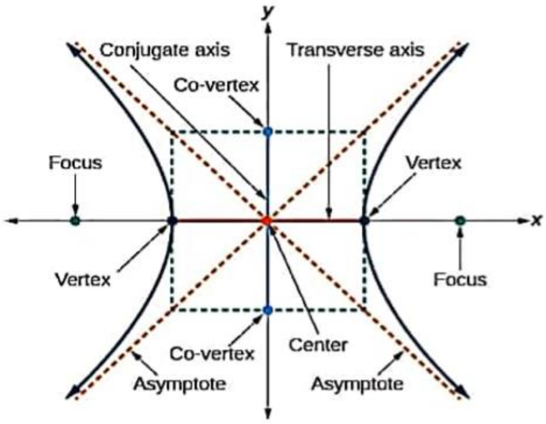
Y = (b/a)x and Y = -(b/a)x



<u>**Eccentricity**</u>

Any branch of a hyperbola can be defined as a curve where the distances of any point from

- A fixed point (the focus) and
- A fixed straight line (the direction) are always in the same ratio



The eccentricity which is the ratio is always greater than 1

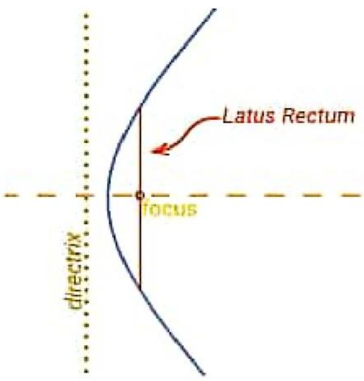
$$E = \sqrt{\frac{a^2 + b^2}{a^2}}$$

Latus Rectum

The latus rectum is the line through the focus and parallel to the directrix. The length of the latus rectum is

$$= \frac{2b^2}{a}$$

The reciprocal function y = 1/x is a hyperbola



Example 3

Identify the vertices and foci of the hyperbola with equation:

$$\frac{y^2}{49} - \frac{x^2}{32} = 1$$

<u>Solution</u>

$$\frac{y^2}{49} - \frac{x^2}{32} = 1$$

It implies that the general form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Resembles the given equation

The traverse axis lies on the y axis. The hyperbola is centred at the origin, so the vertices serve as the y intercepts of the graph.

To find the vertices, set x = 0, and solve for y

$$1 = \frac{y^2}{49} - \frac{x^2}{32}$$

$$1 = \frac{y^2}{49} - \frac{0^2}{32} = \frac{y^2}{49}$$

$y^2 /49 = y = \sqrt{49} = \pm 7$

$$a = \sqrt{32} = 5.656, \qquad b = \sqrt{49} = 7$$

Therefore the foci located at (0, ±c) and solving for c, we have –

$c = \sqrt{a^2 + b^2} = \sqrt{49+32} = \sqrt{81}$

c=9

Therefore, the vertices are located at (0, ±7), Co-vertices at $(0, \pm 5.656)$ and the foci at $(0, \pm 9)$

Example 4

Given is the hyperbola $4x^2 - 9y^2 = 36$ determine the semi-axes, equaton of the asymptotes coordinates of the foci, the eccentricity and the semi-latus rectum.

<u>Solution</u>

$$4x^2 - 9y^2 = 36$$

Divide the equation by 36

$$\frac{4x^2}{36} - \frac{9y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

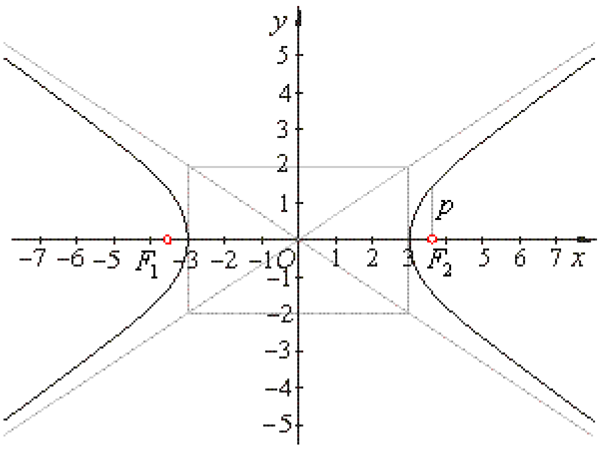
$$a = \sqrt{9} = 3, \qquad b = \sqrt{4} = 2$$

$$Asymptotes \qquad y = \pm\frac{b}{a}x \quad y = \pm\frac{2}{3}x$$

$$Finding\ the\ foci \quad c = \sqrt{a^2 + b^2} == \sqrt{3^2 + 2^2} = \sqrt{13}$$

Therefore the foci located at (0, ±c) $(\pm\sqrt{13}, 0)$

Coordinates of foci, F1 $(-\sqrt{13}, 0)$ F2 $(+\sqrt{13}, 0)$



The eccentricity $e = \frac{c}{a} = \frac{\sqrt{13}}{3}$ and The semi- latus rectum $p = \frac{b^2}{a} = \frac{4}{3}$

Example 5

Write equation of a hyperbola with the focus at F2 (5,0) and whose asymptotes are,

$$y = \pm\frac{3}{4}x$$

<u>Solution</u>

Using F$_2$ (c, 0) and $c = \sqrt{a^2 + b^2}$ $5 = \sqrt{a^2 + b^2}$

$$25 = a^2 + b^2 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots eqn(1)$$

And $y = \pm \frac{b}{a}x$ $y = \pm \frac{3}{4}x \ldots \ldots \ldots \ldots \ldots eqn(2)$

$$Therefore \ \frac{b}{a} = \frac{3}{4} \quad b = \frac{3}{4}a$$

Substituting b in eqn 1

$25 = a^2 + (\frac{3}{4}a)^2$ $25 = \frac{9}{16}a^2 + a^2$ $25 = a^2(\frac{9}{16} + 1)$ $25 = a^2\frac{25}{16}$

$$a^2 = 16 \quad a = 4$$

$$b = \frac{3}{4}a = \frac{3}{4}.4 = 3 \quad b = 3$$

$$a = 4 \quad and \quad b = 3$$

Therefore the equation of the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

Exercises:

a) Identify the foci and the vertices of the hyperbola with the equation

$$\frac{y^2}{25} - \frac{x^2}{9} = 1$$

b) Determine the values of the two foci and two vertices of the hyperbola

$$\frac{x^2}{6} - \frac{y^2}{36} = 1$$

References

- Mathsisfun.com (copyright 2016)
- Luren College Algebra web page
- http://www.nabla.hr/CS-Hyperbola3.htm

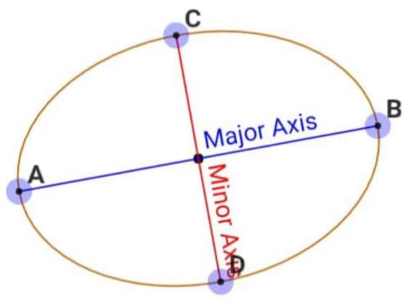
Ellipse

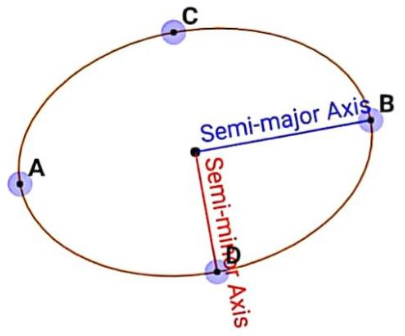
An ellipse is a set of all point on a plane whose distance from two fixed points F and G add up to a constant.

The ellipse has a major axis which is its longest diameter, passing through the center from one side of the ellipse to another side.

The minor axis is the shortest diameter (at the narrowest part of the ellipse)

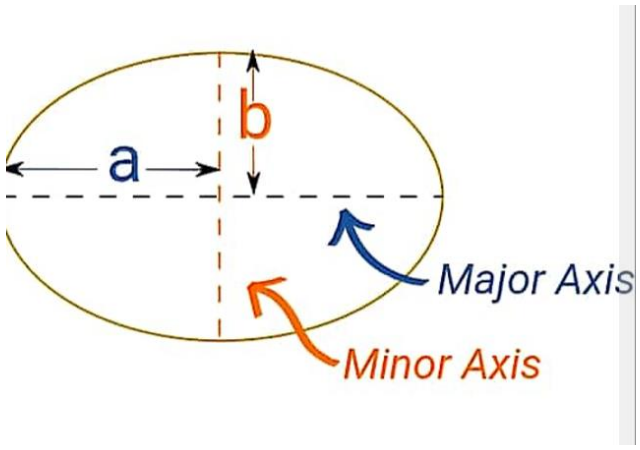
The semi-major axis and the semi minor axis are halves of the original axis.





Formulas in an ellipse

- Area; π x a x b where a is the length of the semi-major axis, and b is that of the semi minor axis



- Perimeter approximation ;

$$2\pi\sqrt{\frac{(a^2+b^2)}{2}}$$

- Eccentricity ; it is a measure of how round the ellipse is, given by

$$e=\frac{\sqrt{a^2-b^2}}{a}$$

- Equation of the ellipse;

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

- Also, from parametric equations x and y can be calculated viz
  x = acos(t)
  y= bsin(t)

Examples6:

Find the equation of an ellipse centered at the origin with major axis of length 10 lying along the x-axis and minor axis of length 6 along the y-axis.

<u>Solution</u>
The major axis has length 10 along the x-axis and is centered at (0,0), so its endpoints are at

$$a=\pm\frac{10}{2}=\pm 5 \qquad b=\pm\frac{6}{2}=\pm 3$$

(-5, 0) and (5,0). Thus, a = 5. Likewise, b = 3. So the equation of this ellipse is:

$$\frac{x^2}{5^2}+\frac{y^2}{3^2}=1 \qquad \frac{x^2}{25}+\frac{y^2}{9}=1$$

Examples7:

Describe the curve represented by $x^2 + 9y^2 - 4x - 72y + 139 = 0$.
Solution:

Collect together the terms involving x and those involving y and complete the Square
(Remember that you are looking for things like $(x - h)^2$ and $(y - k)^2$).

$$x^2 + 9y^2 - 4\text{x} - 72\text{y} + 139 = 0$$

$$(x^2 - 4\text{x}) + (9y^2 - 72\text{y}) + 139 = 0$$
$$(x^2 - 4\text{x}) + 9(y^2 - 8\text{y}) + 139 = 0$$

Add and subtract half of the square coefficient of x and y from the equation

$$Corficient\ of\ x = 4 \quad (\tfrac{1}{2}\times 4)^2 = 4 \qquad Corficient\ of\ y = 8 \quad (8\times\tfrac{1}{2})^2 = 16$$

Thus Add and Subtract 16 and 32 from the equation

$$(x^2 - 4\text{x} + 4 - 4) + 9(y^2 - 8\text{y} + 16 - 16) + 139 = 0$$
$$(x^2 - 4\text{x} + 4) - 4 + 9(y^2 - 8\text{y} + 16) - 9(16) + 139 = 0$$
$$(x^2 - 2\text{x} - 2\text{x} + 4) + 9(y^2 - 4\text{y} - 4\text{y} + 16) - 4 - 144 + 139 = 0$$

$$(x-2)^2 + 9(y-4)^2 - 9 = 0$$
$$(x-2)^2 + 9(y-4)^2 = 9$$

Dividing the equation by 9

$$\frac{(x-2)^2}{9} + \frac{9(y-4)^2}{9} = \frac{9}{9}$$
$$\frac{(x-2)^2}{9} + \frac{(y-4)^2}{1} = 1$$

So, (h, k) = (2, 4), $a = \sqrt{9} = 3$ and $b = 1$

Therefore, the equation represents an ellipse centered at (2, 4) with major axis parallel to the x-axis of length$(3 \times 2) = 6$ and minor axis parallel to the y-axis of length ((1×2) =2.

Reference

- Mathisfun.com
- https://math.usask.ca/emr/examples/ell.html

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